Infinite Divisibility in
Euclidean Quantum Mechanics

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Abstract

In simple – but selected – quantum systems, the probability distribution determined by the ground state wave function is infinitely divisible. Like all simple quantum systems, the Euclidean temporal extension leads to a system that involves a stochastic variable and which can be characterized by a probability distribution on continuous paths. The restriction of the latter distribution to sharp time expectations recovers the infinitely divisible behavior of the ground state probability distribution, and the question is raised whether or not the temporally extended probability distribution retains the property of being infinitely divisible. A similar question extended to a quantum field theory relates to whether or not such systems would have nontrivial scattering behavior.

Introduction, Discussion & Proposition

Preliminary details

An interesting and well studied class of quantum mechanical Hamiltonians consists of examples of the form

$$\mathcal{H} = -\tfrac{1}{2}\partial^2/\partial x^2 + V(x) \;,$$

expressed in units where $m = \hbar = 1$, where the real potential $V(x)$ is chosen so that the spectrum of $\mathcal{H}$ is nonnegative and the real, normalizable, nowhere

vanishing ground state $\phi_o(x)$ has zero energy eigenvalue; such systems are referred to as "simple systems" in this article. In this case, the ground state itself determines the Hamiltonian completely since

$$\mathcal{H} = -\tfrac{1}{2}[\partial^2/\partial x^2 - \phi_o''(x)/\phi_o(x)] \ ,$$

and therefore $\phi_o(x)$ – just as much as $\mathcal{H}$ itself – can be said to determine all the properties of the system.

Without loss of generality we can assume that $\phi_0(x)$ is normalized in the sense that

$$\int \phi_0(x)^2 \, dx = 1 \ ;$$

all integrals without indicated limits of integration extend from $-\infty$ to $+\infty$. In that case, $\phi_0(x)^2$ determines a probability density, and, for all real s, the expression

$$C(s) \equiv \int e^{isx} \phi_0(x)^2 \, dx$$

defines the associated characteristic function. The information contained in the characteristic function $C(s)$ determines the probability density $\phi_0(x)^2$ and thereby the ground state $\phi_0(x)$ itself. Therefore, we can assert that the function $C(s)$ determines all the properties of the system.

The distributions in an interesting *subclass* of all probability distributions have the property of being *infinitely divisible* [1]. An infinitely divisible distribution may be characterized as follows: If X denotes a random variable the characteristic function of which is given by

$$\langle e^{isX} \rangle \equiv C(s) = \int e^{isx} \phi_0(x)^2 \, dx \ ,$$

then, for every integer $N \geq 2$, there exists N independent identically distributed random variables, $Y_n^{(N)}$, $1 \leq n \leq N$, such that

$$X = \sum_{n=1}^{N} Y_n^{(N)} \ .$$

Alternatively stated, this property implies that the Nth root of the characteristic function $C(s)$ is again a characteristic function, i.e.,

$$C_{(N)}(s) \equiv C^{1/N}(s) = \int e^{isx} \rho_{(N)}(x) \, dx \ ,$$

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where $\rho_{(N)}(x)$ denotes a probability density for all N. Note well that only a subclass of all probability distributions exhibits infinite divisibility.

Remark: Integer powers of characteristic functions always correspond again to characteristic functions. Therefore, for infinitely divisible distributions, the expression $C^{M/N}(s)$ again describes a characteristic functions for all positive rational numbers M/N. All characteristic functions are continuous, and it follows that in the limit as those rational numbers tend to an arbitrary positive real number, i.e., a limit such that $M/N \rightarrow r$, where $0 < r < \infty$, the result is again a characteristic function defined by $C^r(s)$.

The general form of characteristic functions which are infinitely divisible is well known [1]. For simplicity we shall confine attention to only those distributions which are *even*, i.e., for which $V(-x) = V(x)$, leading to an even ground state $\phi_0(-x) = \phi_0(x)$, and thereby to an even characteristic function $C(-s) = C(s)$. In that case, the characteristic function $C(s)$ of an even, infinitely divisible distribution may be represented in the form [1]

$$C(s) = \exp\{-\tfrac{1}{2}as^2 - \int[1 - \cos(sy)]\,\sigma(y)\,dy\}\,,$$

where $a \geq 0$ and $\sigma(y) \geq 0$ for all y; clearly, $\sigma(-y) = \sigma(y)$. In this expression we have allowed ourselves one simplification: namely, we have chosen an absolutely continuous measure $\sigma(y)\,dy$. Existence of this expression requires that

$$\int[y^2/(1+y^2)]\sigma(y)\,dy < \infty\,.$$

However, it may well happen that

$$\int\sigma(y)\,dy = \infty\,,$$

and indeed this situation is relatively common, especially in our studies.

The general form of the characteristic function for infinitely divisible distributions implies that the associated random variable X, for which $\langle e^{isX} \rangle = C(s)$, may be decomposed into a sum of *two, independent* random variables, $X = X_G + X_P$, where the random variable X_G has a Gaussian distribution (determined by a) and the random variable X_P has a Poisson distribution (determined by σ); more precisely, X_P has a compound Poisson distribution or a generalized Poisson distribution depending on whether $\int\sigma(y)\,dy$ is finite or infinite, respectively [2].

It is noteworthy that $\ln(C(s))$ is the generator of the truncated (= connected) moments of the distribution. Therefore, for infinitely divisible distributions, it follows that

$$\langle (e^{isX} - 1)^T \rangle = -\tfrac{1}{2} a s^2 - \int [1 - \cos(sy)] \, \sigma(y) \, dy \, .$$

Assuming the necessary moments exist, we learn that the truncated moments (superscript T) are given by

$$\langle X^{2p} \rangle^T = a \delta_{p1} + \int y^{2p} \sigma(y) \, dy \, ,$$

and as a consequence the even order truncated moments are always nonnegative. This will be an important feature in our futher investigations.

We shall not be interested in examples that have both a Gaussian and a Poisson contribution. Indeed, we shall focus on the wide class of examples that arise from Poisson distributions alone and thus we assume that there is *no* Gaussian contribution. This means we shall set $X_G = 0$ or equivalently assume that the parameter a vanishes. Having said this, we are left to focus on that subclass of characteristic functions which can be written in the form

$$C(s) = \exp\{ -\int [1 - \cos(sy)] \, \sigma(y) \, dy \} \, ,$$

for nonnegative functions σ such that

$$\int [y^2/(1 + y^2)] \, \sigma(y) \, dy < \infty \, .$$

We shall also write

$$U(y)^2 \equiv \sigma(y) \, ,$$

and we shall call the nonnegative function $U(y) \, [= U(-y)]$ the *model function.*

Just as we have asserted that the characteristic function $C(s)$ determines all the physics of our problem, we are clearly able to declare that *the model function $U(y)$ determines all the physics of our problem.* In other words, we can, without loss of generality, adopt the model function $U(y)$ as the primary input defining the problem at hand. Indeed, this view of the problem has the advantage that choosing $U(y)$ initially, and within a certain suitable class of functions, ensures that we are dealing with an even potential $V(x)$ that leads to a ground state $\phi_0(x)$ which in turn generates an infinitely divisible distribution that involves only a Poisson contribution. Of course, starting with the model function is often easier said then done!

To demonstrate that this set of conditions is not empty, it is appropriate to present a few examples.

Examples

Example 1: The first example is an idealized example that has the advantage of being analytically quite simple. It is convenient to initiate our description in terms of the ground state wave function

$$\phi_0(x) = \frac{\sqrt{a}}{\sqrt{\pi(a^2 + x^2)}} \ ,$$

where $a > 0$ is a fixed parameter. In turn, this example corresponds to the potential

$$V(x) = \frac{2x^2 - a^2}{(a^2 + x^2)^2} \ .$$

The characteristic function appropriate to this example is given by

$$
\begin{aligned}
C(s) &= \int \frac{a\,e^{isx}}{\pi(a^2 + x^2)}\,dx \\
&= \exp(-a|s|) \\
&= \exp\{-a\int[1 - \cos(sy)](1/\pi y^2)\,dy\} \ .
\end{aligned}
$$

Thus, for this example, we see that the model function reads

$$U(y) = \frac{\sqrt{a}}{\sqrt{\pi}} \frac{1}{|y|} \ .$$

It is true that $\int U(y)^2\,dy = \infty$, due to a divergence at $y = 0$. However, the present distribution falls very slowly as $y \to \infty$, so slowly in fact that no moments of the distribution exist.

The next example is similar to the present one except that the moments of the distribution all exist.

Example 2: For this model we start with the characteristic function of the ground state probability density in the form

$$
\begin{aligned}
C(s) = e^{-b\sqrt{s^2 + \rho^2} + b\rho} &= \int e^{isx} \frac{b\rho}{\pi} \frac{K_1(\rho\sqrt{x^2 + b^2})}{\sqrt{x^2 + b^2}} e^{b\rho}\,dx \\
&= \exp\{-b\int[1 - \cos(sy)] \frac{\rho K_1(\rho|y|)}{\pi|y|}\,dy\} \ ,
\end{aligned}
$$

where $b > 0$ and $\rho > 0$ are free parameters at our disposal. Here, K_1 denotes the standard Bessel function. The ground state wave function for this example may be read off from the integrand of the characteristic function. In turn, the ground state implicitly determines the potential $V(x)$. Clearly the model function for this case is given by

$$U(y) = \frac{\sqrt{b\rho\, K_1(\rho|y|)}}{\sqrt{\pi\, |y|}}\ .$$

Since $K_1(\rho|y|) \simeq 1/(\rho|y|)$ for small argument, it follows that *near the singularity* at $y = 0$, the behavior of $U(y)$ is actually identical in the two examples when $b = a$; indeed, in the limit that $\rho \to 0$, it follows that Example 2 reduces to Example 1.

As a vast generalization of both Examples 1 and 2, we may consider characteristic functions of the general form

$$\exp\{-b\int [1 - \cos(sy)]\, F(y)/(\pi y^2)\, dy\} \equiv \int e^{isx}\, G(x)\, dx \ ,$$

generated by basic functions $F \in C^2$ with $F(0) = 1$, $F \geq 0$, and $\int [F(y)/(1 + y^2)]\, dy < \infty$. Generally speaking, after being given an analytic expression for the basic function for such examples, we cannot analytically specify the L^1 nonnegative weight function G describing the ground state probability density; however, we know that such a probability density exists.

Euclidean time dependence

The Hamiltonian for our system can be used to propagate the time forward either in real time by the evolution operator $\exp(-i\mathcal{H}T)$ or in imaginary time by the evolution operator $\exp(-\mathcal{H}T)$. The latter case corresponds to Euclidean quantum mechanics, which as is well known, can be described by a stochastic process involving a stochastic variable $X(t)$, $-\infty < t < \infty$, with a distribution determined by a Feynman-Kac like probability distribution [3]. Expanding the meaning of the symbols representing expectation values, i.e., $\langle\, (\cdot)\, \rangle$, to cover the stochastic variable $X(t)$, we are led to consider correlation functions of the form

$$\int \langle X(t_1)\, X(t_2) \cdots X(t_p) \rangle\, f_p(t_1, t_2, \ldots, t_p)\, dt_1\, dt_2 \cdots dt_p \ ,$$

for all $p \geq 1$ and all suitable weight functions f_p. In turn, correlation functions may alternatively be described by a suitable generating functional having the meaning of a characteristic functional. Specifically, we have in mind the expression

$$E\{u\} \equiv \langle \exp[i \int u(t) X(t)\, dt] \rangle \, ,$$

defined for all smooth test functions u. According to the tenants of Euclidean quantum mechanics, the functional $E\{u\}$ obeys all the appropriate positive-definite inequalities and continuity to satisfy the Bochner-Minlos axioms to be the functional Fourier transform of a suitable probability distribution [4].

We further observe that since the Hamiltonian is explicitly time independent, the associated stochastic process is stationary. Specifically this implies a time translation invariance of the characteristic functional, which takes the form

$$\langle \exp[i \int u(t-\tau) X(t)\, dt] \rangle = \langle \exp[i \int u(t) X(t)\, dt] \rangle$$

for any τ, $-\infty < \tau < \infty$.

There is a connection of the characteristic functional $E\{u\}$ involving all time and the characteristic function $C(s)$ introduced above that occurs at any one time, say at time $t = 0$. If we choose the function $u(t)$ that enters the characteristic functional as

$$u(t) = s\, \delta(t) \, ,$$

where $\delta(t)$ is a Dirac delta function, then is follows that

$$E\{u(\cdot) = s\, \delta(\cdot)\} = C(s) \, ;$$

hence, the restricted evaluation of the characteristic functional E coincides with the characteristic function C defined at a single time.

We can make a similar connection with any one of the correlation functions themselves. For example, let us focus on

$$\int \langle X(t_1) \cdots X(t_4) \rangle \, f_4(t_1, \ldots, t_4)\, dt_1 \cdots dt_4$$

evaluated for

$$f_4(t_1, \ldots, t_4) = \delta(t_1) \cdots \delta(t_4) \, ,$$

leading to $\langle X(0)^4 \rangle$. We assert the equality given by

$$\langle X(0)^4 \rangle = \int x^4 \, \phi_0(x)^2 \, dx \, ,$$

i.e., the fourth moment of the ground-state probability distribution.

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Now we come to the whole point of the present paper!

By choice, we have restricted the sharp-time, ground-state probability distribution to be infinitely divisible. This is encoded into the fact that an arbitrary, positive fractional power of the characteristic function is again a characteristic function. Since the full-time characteristic functional (E) collapses to the sharp-time characteristic function (C) when the testing functions are evaluated at a sharp time (say, $t = 0$), it follows *for such a set of limited test functions* that the sharp-time restricted characteristic functional is infinitely divisible. The question arises, therefore, *whether the FULL time characteristic functional is itself infinitely divisible as well*, or whether that property is restricted to those stochastic variables with a time spread restricted to a window of finite size. It may also happen that certain distributions enjoy full time infinite divisibility following from sharp time infinite divisibility, while other distributions do not exhibit full time infinite divisibility even though they have sharp time infinite divisibility. It is clear that the question raised here is fairly complicated. The simplest possibility would seem to be to show that sharp time infinite divisibility does not imply full time infinite divisibility, and this could be determined by finding just one example for which a single property required by infinite divisibility fails to hold.

Selection of the test question

The structure of infinitely divisible distributions is such that the *truncated* correlation functions are positive definite functions. In particular, for such distributions, it follows that

$$\langle (\smallint f(t) X(t)\, dt)^4 \rangle^T \equiv \langle (\smallint f(t) X(t)\, dt)^4 \rangle - 3\langle (\smallint f(t) X(t)\, dt)^2 \rangle^2 > 0$$

unless $f = 0$. The simplest example derives from a uniform weighting where $f(t) = 1$ in an interval $-T < t < T$, and $f(t)$ is zero elsewhere. We should like to examine the behavior of this truncated correlation function for asymptotically large T, and compare it with the behavior for extremely small T. Due to stationary of the underlying ensemble, the large T behavior diverges as $2T$; for small times, on the other hand, it behaves as $(2T)^4$ times the sharp time expectation value. To eliminate these unimportant temporal factors, it is convenient to rescale the truncated four point function by a factor

$(1 + 2T)^3/(2T)^4$, which leads to the alternative form given by

$$\chi_2 \equiv \frac{(1 + 2T)^3}{(2T)^4} \left[\langle (\int_{-T}^{T} X(t)\, dt)^4 \rangle^T \right]$$

$$= \frac{(1 + 2T)^3}{(2T)^4} \left[\langle (\int_{-T}^{T} X(t)\, dt)^4 \rangle - 3 \langle (\int_{-T}^{T} X(t)\, dt)^2 \rangle^2 \right] .$$

Although this is just one measure of the truncated four point function it is a significant one and one that is comparatively easy to evaluate. If $\chi_2 < 0$ for large T, it would establish that the full time probability functional is *not* infinitely divisible even though the sharp time probability function has been chosen (by design) to be infinitely divisible. Of course, if $\chi_2 > 0$ for all T, then no definitive conclusion can be drawn about the nature of the full time probability distribution. To proceed further, one would need to check other moments, or even change to another model function in order to test the nature of the full time probability distribution. At present, the author is not aware of any general scheme that could decide whether or not the full time probability functional is infinitely divisible whenever the sharp time probability function has been chosen to be infinitely divisible. Nevertheless, in the absence of any proof to the contrary, it seems reasonable to *conjecture* that infinite divisibility for sharp time simple quantum systems does *not* imply full time infinite divisibility for them.

Alternative representation

It is noteworthy that the evaluation of χ_2 can be reexpressed in terms of the eigenfunctions and eigenvalues of the Hamiltonian for our simple system, and we now turn our attention to develping this alternative expression for χ_2. For convenience, let us consider those special cases where $\mathcal{H}$ has a purely discrete, nondegenerate spectrum such that

$$[-\tfrac{1}{2}\partial^2/\partial x^2 + V(x)]\phi_n(x) = E_n \phi_n(x) ,$$

where $0 = E_0 < E_1 < E_2 \cdots$ and all the (real) eigenfunctions form a complete orthonormal set of functions for which

$$\int \phi_n(x)\phi_m(x)\, dx = \delta_{nm} .$$

Here, as before, the ground state is denoted by $\phi_0(x)$. We shall also introduce these relations in the usual abstract bra-ket notation as well, namely

$$[\tfrac{1}{2}P^2 + V(Q)]|n\rangle = E_n |n\rangle ,$$

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with $|n\rangle$ denoting the abstract eigenvectors, for which $\langle n|m\rangle = \delta_{nm}$. Of course, we let $|0\rangle$ denote the ground state.

By definition, the correlation functions are *symmetric* functions of their temporal arguments. However, when expressed in terms of the eigenvectors and eigenvalues, it proves convenient to define the correlation functions in a *time ordered* manner. In particular, for the time ordered two point function, this rule leads to the expression

$$\langle X(t)X(u)\rangle = \sum_{n=0}^{\infty} \langle 0|Q|n\rangle\, e^{-E_n(t-u)}\, \langle n|Q|0\rangle\,, \qquad t \geq u\,;$$

we can of course extend this expression to all t and u values simply by replacing $(t-u)$ by $|t-u|$ on the right hand side, but we choose not to do so in order to simplify the analysis in what follows. It is clearly convenient to introduce the notational shorthand that

$$Q_{kl} \equiv \langle k|Q|l\rangle\,.$$

Similar expressions also exist for higher-order correlation functions. For example, for the time ordered four point function we have

$$\langle X(t)X(u)X(v)X(w)\rangle = \sum_{k,l,m=0}^{\infty} Q_{0k}\, e^{-E_k(t-u)}\, Q_{kl}\, e^{-E_l(u-v)}$$
$$\times\, Q_{lm}\, e^{-E_m(v-w)}\, Q_{m0}\,, \qquad t \geq u \geq v \geq w\,.$$

From the assumed symmetry of the potential, i.e., $V(-x) = V(x)$, it follows that the eigenfunctions have *alternating parity*, namely, that $\phi_n(-x) = (-1)^n \phi_n(x)$. Consequently, the matrix elements of Q, such as $\langle k|Q|l\rangle = Q_{kl}$, only connect eigenvectors of *opposite* parity; in other words, if k is even, then l is odd, or vice versa. In particular, for the two point function, as presented above, only *odd* values of n lead to nonvanishing contributions. For the four point function, as presented above, only odd values of k and m contribute, while only even values of l need be considered.

To begin our construction of χ_2, we may integrate our time ordered expressions over a restricted time region and rescale the result to account for such a limited integration domain. Initially, this recipe implies that

$$\langle (\int_{-T}^{T} X(t)\,dt)^4\rangle = 24 \left[\sum_{k,l,m} Q_{0k}\, Q_{kl}\, Q_{lm}\, Q_{m0} \right] \int_{-T}^{T} dt \int_{-T}^{t} du$$
$$\times \int_{-T}^{u} dv \int_{-T}^{v} dw \exp[-E_k(t-u) - E_l(u-v) - E_m(v-w)]\,.$$

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Here the factor $24 = 4!$ corrects for the limited domain of integration due to time ordering. For small T, the integral above involves terms $O(T^4)$, while for large T, this integral involves terms $O(T^2)$, $O(T)$, and $o(T)$. Since we content ourselves with the integral's value for very small and very large T values, it will suffice to consider terms of the indicated types, respectively.

Small T behavior

First, consider very small T values, or more specifically consider the situation where $E_k T \ll 1$ for all sensibly contributing energy values. In that case, we can replace all exponentials in the previous formula by unity, and therefore, to leading order

$$\frac{(1+2T)^3}{(2T)^4} \langle (\int_{-T}^{T} X(t)\, dt)^4 \rangle = \sum_{k,l,m} Q_{0k}\, Q_{kl}\, Q_{lm}\, Q_{m0}\,, \quad T \ll 1\,.$$

This result of course is equivalent to the sharp time four point moment. Second, consider the square of the two point function, which leads to the result

$$\frac{(1+2T)^3}{(2T)^4} \langle (\int_{-T}^{T} X(t)\, dt)^2 \rangle^2 = \sum_{k,m=1}^{\infty} Q_{0k}\, Q_{k0}\, Q_{0m}\, Q_{m0}\,, \quad T \ll 1\,,$$

to leading order in T. This expression, of course, is just the square of the sharp time second moment. Finally, the truncated four point moment for small T is given by

$$\chi_2 = \sum_{k,l,m=1}^{\infty} Q_{0k}\, Q_{kl}\, Q_{lm}\, Q_{m0} - 2 \sum_{k,m=1}^{\infty} Q_{0k}\, Q_{k0}\, Q_{0m}\, Q_{m0}\,;$$

note well, that for the first term all three sums omit the ground state, and for the second term both sums omit the ground state. This change of the first term has resulted in the factor 3 becoming a factor 2 in the second term.

We can readily check this result for the harmonic oscillator, namely where $V(x) = \frac{1}{2}\omega^2 x^2$. In that case, the only nonvanishing matrix elements of interest are $Q_{01} = Q_{10} = 1/\sqrt{2\omega}$ and $Q_{12} = Q_{21} = 1/\sqrt{\omega}$. Therefore,

$$\chi_2 = Q_{01}\, Q_{12}\, Q_{21}\, Q_{10} - 2\, [Q_{01}\, Q_{10}]^2$$
$$= [1/(2\omega^2) - 2/(4\omega^2)] = 0\,.$$

This is the correct result: Since the ground state for the harmonic oscillator is a Gaussian, the distribution is infinitely divisible; and, as a Gaussian, all truncated moments other than the second moment vanish.

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Large T behavior

Let us now turn our attention to the evaluation of χ_2 for large T. In this case it is useful to divide the fourth moment into two distinct expressions, namely,

$$\frac{(1+2T)^3}{(2T)^4} \langle (\int_{-T}^{T} X(t)\, dt)^4 \rangle = A + B \ .$$

In this expression,

$$A \equiv 24 \sum_{k,l,m=1}^{\infty} Q_{0k} Q_{kl} Q_{lm} Q_{m0} \left[\frac{1}{E_k} \frac{1}{E_l} \frac{1}{E_m} \right] ,$$

which is valid for $T \gg 1$, provided all E values are strictly positive, $E > 0$. This restriction leads to the omission of the term for $l = 0$ above. The omitted term, where $l = 0$, can be evaluated separately, and to leading order it follows that

$$B \equiv 24 \sum_{k,m=1}^{\infty} Q_{0k} Q_{k0} Q_{0m} Q_{m0} \left[T \frac{1}{E_m} \frac{1}{E_k} - \frac{1}{E_k^2} \frac{1}{E_m} - \frac{1}{E_k} \frac{1}{E_m^2} \right] .$$

The two point function for large T follows from the expression given by

$$\langle (\int_{-T}^{T} X(t)\, dt)^2 \rangle = \sum_{m=1}^{\infty} Q_{0k} Q_{k0} \left[\frac{4T}{E_k} - \frac{2}{E_k^2} \right] .$$

Consequently, for very large T, we observe that

$$\frac{(1+2T)^3}{(2T)^4} \left(\langle (\int_{-T}^{T} X(t)\, dt)^2 \rangle \right)^2$$

$$= \frac{1}{2T} \sum_{k,m=1}^{\infty} Q_{0k} Q_{k0} Q_{0m} Q_{m0} \left[\frac{4T}{E_k} - \frac{2}{E_k^2} \right] \left[\frac{4T}{E_m} - \frac{2}{E_m^2} \right]$$

Finally, for large T,

$$\frac{(1+2T)^3}{(2T)^4} \langle (\int_{-T}^{T} X(t)\, dt)^4 \rangle^T$$

$$= 24 \sum_{k,l,m=1}^{\infty} Q_{0k} Q_{kl} Q_{lm} Q_{m0} \left[\frac{1}{E_k} \frac{1}{E_l} \frac{1}{E_m} \right]$$

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$$+24\,T\sum_{k,m=1}^{\infty}Q_{0k}\,Q_{k0}\,Q_{0m}\,Q_{m0}\left[\frac{1}{E_k}\frac{1}{E_m}\right]$$

$$-24\sum_{k,m=1}^{\infty}Q_{0k}\,Q_{k0}\,Q_{0m}\,Q_{m0}\left[\frac{1}{E_k}\frac{1}{E_m^2}+\frac{1}{E_k^2}\frac{1}{E_m}\right]$$

$$-3\,\frac{1}{2T}\sum_{k,m=1}^{\infty}Q_{0k}\,Q_{k0}\,Q_{0m}\,Q_{m0}\left[\frac{4T}{E_k}-\frac{2}{E_k^2}\right]\left[\frac{4T}{E_m}-\frac{2}{E_m^2}\right]\,,$$

which to leading order in T becomes

$$\chi_2=24\sum_{k,l,m=1}^{\infty}Q_{0k}\,Q_{kl}\,Q_{lm}\,Q_{l0}\left[\frac{1}{E_k}\frac{1}{E_l}\frac{1}{E_m}\right]$$

$$-24\sum_{k,m=1}^{\infty}Q_{0k}\,Q_{k0}\,Q_{0m}\,Q_{m0}\left[\frac{1}{E_k}\frac{1}{E_m^2}\right]\,.$$

Again, it is useful to test this expression with the harmonic oscillator. Besides the matrix elements already given previously, we need the first two excited state energy levels, $E_1=\omega$ and $E_2=2\omega$. In that case, we find that

$$\chi_2=24\left[(1/2\omega^2)\cdot(1/2\omega^3)-(1/4\omega^2)\cdot(1/\omega^3)\right]=0\,,$$

as expected.

Summary of principal formulas

The truncated four point function given by

$$\chi_2=(1+2T)^3/(2T)^4\,\langle(\int_{-T}^{T}X(t)\,dt)^4\rangle^T$$

is expressed alternatively by

$$\chi_2=\sum_{k,l,m=1}^{\infty}Q_{0k}Q_{kl}Q_{lm}Q_{m0}-2\sum_{k,m=1}^{\infty}Q_{0k}Q_{k0}Q_{0m}Q_{m0}\,,$$

for *small* T, and by

$$\chi_2=24\sum_{k,l,m=1}^{\infty}Q_{0k}\,Q_{kl}\,Q_{lm}\,Q_{l0}\left[\frac{1}{E_k}\frac{1}{E_l}\frac{1}{E_m}\right]$$

$$-24\sum_{k,m=1}^{\infty}Q_{0k}\,Q_{k0}\,Q_{0m}\,Q_{m0}\left[\frac{1}{E_k}\frac{1}{E_m^2}\right]\,,$$

for *large T*. In this expression, $Q_{kl} \equiv \langle k|Q|l \rangle$ denotes a matrix element of the position operator Q, where the states $|k\rangle$ are normalized eigenvectors of the time-independent Schrödinger equation,

$$[\tfrac{1}{2}P^2 + V(Q)]\,|k\rangle = E_k\,|k\rangle \;,$$

for the real, even potential, $V(Q)$.

Remark: The apparent difference in dimensions between the behavior for small and large T arises from our choice of a scaling factor; had we used

$$(1 + 2ET)^3/(2ET)^4$$

instead of $(1 + 2T)^3/(2T)^4$, where E is some characteristic energy level, then the two extremal expressions would have had the same dimensions. However, since our main concern is to study simple systems for which $\chi_2 > 0$ for small T with the aim of finding examples for which $\chi_2 < 0$ for large T, the formal difference in dimensionality is of little concern.

Why bother?

The reader may well ask why should one care about quantum systems that have infinitely divisible distributions for sharp time position variables and may – or may not – have infinitely divisible distributions for full time position variables. A brief explanation may help clarify the situation.

A Euclidean formulation for a scalar quantum field theory is characterized by a stochastic field variable we may call $\phi(x, t)$. The statistics of this variable are governed by a probability distribution, which, like our single degree of freedom examples discussed above, may be described by a characteristic functional

$$\left\langle e^{i\int u(x,t)\,\phi(x,t)\,dx\,dt} \right\rangle .$$

The sharp time expression is given simply by setting $u(x, t) = v(x)\delta(t)$ leading to

$$\left\langle e^{i\int v(x)\,\phi(x,0)\,dx} \right\rangle .$$

Interacting quantum field theories encounter divergences, and this is as true for the sharp time field expressions as it is for the full time field expressions. In a recent paper [5] it was argued that sharp time formulations

involving an infinitely divisible field distribution tended to alleviate some of the principal causes of field theory divergences. From this point of view there would seem to be some merit in seeking to formulate matters so that sharp time fields had infinitely divisible distributions. Accepting such an argument opens the question of whether an infinitely divisible distribution for sharp time fields does – or does not – force the full time field distribution to be infinitely divisible.

Suppose, for the sake of argument, it was true that the full time field distribution was also infinitely divisible. By an argument of Buchholz and Yngvason [6] this would result in a theory with a unit scattering matrix. For example, consider the truncated four point function

$$\langle \phi(f_1)\phi(f_2)\phi(g_2)\phi(g_1)\rangle^T ,$$

where

$$\phi(f) \equiv \int f(x,t)\phi(x,t)\, dx\, dt ,$$

for f a smooth test function, etc. For infinitely divisible distributions certain truncated correlation functions are nonnegative, such as

$$0 \leq \langle \phi(g_1)\phi(g_2)\phi(g_2)\phi(g_1)\rangle^T .$$

Consequently, by the Schwarz inequality,

$$0 \leq |\langle \phi(f_1)\phi(f_2)\phi(g_2)\phi(g_1)\rangle^T|^2$$
$$\leq \langle \phi(f_1)\phi(f_2)\phi(f_2)\phi(f_1)\rangle^T \langle \phi(g_1)\phi(g_2)\phi(g_2)\phi(g_1)\rangle^T .$$

Now, take suitable limits so that

$$\phi(g_1) \rightarrow \phi_{in}(g_1) , \qquad \phi(g_2) \rightarrow \phi_{in}(g_2) ,$$
$$\phi(f_1) \rightarrow \phi_{out}(f_1) , \qquad \phi(f_2) \rightarrow \phi_{out}(f_2) ,$$

where "in" and "out" fields denote free fields appropriate to the asymptotic regime. In that case we find

$$0 \leq |\langle \phi_{out}(f_1)\phi_{out}(f_2)\phi_{in}(g_2)\phi_{in}(g_1)\rangle^T|^2$$
$$\leq \langle \phi_{out}(f_1)\phi_{out}(f_2)\phi_{out}(f_2)\phi_{out}(f_1)\rangle^T$$
$$\times \langle \phi_{in}(g_1)\phi_{in}(g_2)\phi_{in}(g_2)\phi_{in}(g_1)\rangle^T = 0 ,$$

leading to no two particle scattering. Extending this type of argument leads to a trivial theory in the sense that the scattering matrix $S = \mathbb{1}$.

If, however, an infinitely divisible sharp time field distribution did *not* imply that the full time field distribution was infinitely divisible, then triviality of the scattering matrix may be avoided.

This kind of question is extremely hard to study for a field theory. Hence, the study initiated in this paper is restricted to a single degree of freedom quantum system with the Hamiltonian form chosen to be similar in spirit to that of a traditional scalar field theory. If a positive outcome of the single degree of freedom problem emerges, it may well suggest that further study of the quantum field situation may be worthwhile.

A proposed problem

As a simple examination of the associated ground state reveals, neither of the two specific examples (Examples 1 & 2) suggested above are suitable for a detailed study since their spectrum is part discrete and part continuous. Unfortunately, distribution functions for general infinitely divisible characteristic functions are known for very few examples [1]. We conjecture that certain examples, such as

$$
\begin{aligned}
C(s) &= \exp\{-\int [1 - \cos(sy)]\, e^{-y^2}/(\pi\,|y|^\alpha)\, dy\} \\
&\equiv \int \cos(sx)\, \phi_0(x)^2\, dx \ ,
\end{aligned}
$$

for $2 \leq \alpha < 3$, may correspond to potentials with a purely discrete spectrum which would then permit our formulas to be applied.

The implicitly defined ground state $\phi_0(x)$ above may be determined numerically. If the ground state falls to zero suitably faster than an exponential, then the spectrum of the Hamiltonian would be purely discrete. In turn, the potential associated with this ground state is

$$
V(x) \equiv \phi_0''(x)/2\,\phi_0(x) \ ,
$$

which again could be numerically determined. Even if there are small errors in the numerical determination, it is quite likely that the simple system still lies within the special class that is infinitely divisible for the ground state distribution. With the potential $V(x)$ – with symmetry enforced – now determined, standard computer programs could be used to calculate a

number of eigenfunctions and eigenvalues, as well as suitable matrix elements. These quantities permit the calculation of χ_2 for large and small T values. If, for large T, $\chi_2 < 0$, the desired result will have been achieved; if, instead, $\chi_2 > 0$, then further investigations are warranted.

Of course, it may be true that sharp time infinite divisibility necessarily implies full time infinite divisibility. If this implication could be proved, then that would be an important result which would make any computation (such as outlined in the previous paragraph) completely unnecessary.

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